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Time Operators and Approximation of Continuous Functions

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Abstract The main purpose of this paper is to study time operators associated with generalized shifts and determined by the Haar and Faber–Schauder bases on the space of continuous functions. It is given the characterization of the domains of the constructed time operators and their scalings. It is also shown how scalings of time operators affect the dynamics of associated semigroups of shift operators.

Keywords Time operator · Evolution semigroup · Shift operator · Haar basis · Faber–Schauder basis

1 Introduction

Time operators play an important role in the theory of irreversibility in classical physics. They are also a useful tool in the spectral analysis and prediction in those dynamical systems where the time evolution can be formulated in terms of semigroups of operators on vector spaces. For this reason there is a growing interest in expanding the methods of analysis based on time operators which were initially associated with a relatively narrow class of reversible dynamical systems.

The idea behind time operator is to connect a unitary group of evolution operators $\{V_t\}$ with a selfadjoint operator T through the commutation relation

$$V_t^{-1} T V_t = T + tI, \tag{1}$$

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with I the identity operator. In physics, the group $\{V_t\}$ describes a (reversible) time evolution of a distinguished class of elements ρ of the space $\mathcal{H}$ called the *states*

$$\rho \quad \longmapsto \quad \rho_t \stackrel{\mathrm{df}}{=} V_t \rho.$$

For each state ρ from the domain of T the scalar product $(\rho, T\rho)$ represents the average *age* of ρ and relation (1) assumes the form

$$(\rho_t, T\rho_t) = (\rho, T\rho) + t\|\rho\|^2. \tag{2}$$

The physical meaning of (2) is that the average age of each evolved state keeps step with the external clock time.

In classical dynamical systems time operators have been initially constructed for K-systems and K-flows (see [13, 15]). A characteristic feature of these reversible dynamical systems is a very high instability. The observed time evolution more resembles stochastic than reversible. The main purpose of introducing time operators into K-systems was an attempt to resolve the old standing physical problem of reversibility versus irreversibility [15]. It turned out that by scaling the age eigenvalues, i.e. changing the internal time of a dynamical system, it is possible to obtain different kinds of time evolutions. In particular, it is possible to reconcile the invertible unitary evolution of unstable dynamical systems with observed Markovian evolution approaching to equilibrium [14]. The unitary dynamics on a Hilbert space that admits a time operator is called intrinsically irreversible. The question of existence of time operators for unitary groups is thoroughly tackled in [9].

It has been found recently [1–3, 18, 19] that time operators can be also associated with non-invertible dynamics and used as a new tool in the spectral analysis of evolution semigroups of unstable dynamical systems. Namely, an operator T on a Hilbert space $\mathcal{H}$ is called a time operator associated with a semigroup $\{V_t\}$ of partial isometries on $\mathcal{H}$ if the following relation holds

$$TV_t = V_t T + tV_t. \tag{3}$$

Then idea behind the spectral analysis of the evolution semigroup $\{V_t\}$ through the time operator T is to decompose T in terms of its eigenvectors $\varphi_{n,k}$, $T\varphi_{n,k} = n\varphi_{n,k}$

$$T = \sum_n n \sum_k (\cdot, \varphi_{n,k})\varphi_{n,k}$$

in such a way that the system $\{\varphi_{n,k}\}$ is complete in $\mathcal{H}$, i.e. $\sum_{n,k}(\cdot, \varphi_{n,k})\varphi_{n,k} = I$, and the operator V_t shifts the eigenvectors $\varphi_{n,k}$

$$V_t\varphi_{n,k} = \varphi_{n+t,k}$$

(the index n labels the age and k the multiplicity of the spectrum of the time operator). As a result the eigenvectors $\varphi_{n,k}$ of the time operator provide a shift representation of the evolution

$$f = \sum_{n,k} a_{n,k}\varphi_{n,k} \quad \Longrightarrow \quad V_t f = \sum_{n,k} a_{n,k}\varphi_{n+t,k} = \sum_{n,k} a_{n-t,k}\varphi_{n,k}.$$

The knowledge of the eigenvectors of T amounts therefore to solving the prediction problem for the dynamical system described by the semigroup $\{V_t\}$. The spaces $\mathcal{H}_n$ spanned by the eigenvectors $\varphi_{n,k}$ are called the age eigenspaces or the spaces of innovations at time n, as they correspond to the new information or detail brought at time n. The analysis of evolution semigroups based on a time operator is called the *time operator method*.

The simplest links between time operators and approximation theory can be obtained through wavelets. An arbitrary wavelet multiresolution analysis can be viewed as a K-system determining a time operator whose age eigenspaces are the wavelet detail subspaces. Conversely the eigenspaces of the time operator can be expanded from the unit interval to the real line giving the multiresolution analysis corresponding to the Haar wavelet [1]. In [2] the reader can also find connections between time operators and the Shannon–Nyquist theorem.

Time operators are usually defined on Hilbert spaces. However, there are also other important vector spaces associated with approximation. Such are the Banach spaces L^p, $p \geq 1$, or $\mathcal{C}_{[a,b]}$—the space of continuous functions on an interval $[a,b]$. The space of continuous functions plays also a major role in the study of trajectories of stochastic processes.

Time operator can be, in principle, defined on a Banach space in the same way as on a Hilbert space although there are some essential differences. For example, a given nested family of closed subspaces of a Hilbert space determines a self adjoint operator with the spectral projectors onto those subspaces. This is not true in arbitrary Banach space because it is not always possible to construct an analog of the orthogonal projectors on closed subspaces [12].

For some dynamical systems associated with maps time operators have been extended from the Hilbert space L^2 to the Banach space L^p by replacing the methods of spectral theory [8, 14], by more efficient martingales methods [16, 17]. For example, for a K-flow it is possible to extend the time operator from L^2 to L^1 in such a way that its domain contains absolutely continuous measures on the phase space.

In this paper we construct the time operators associated with the Haar and the Faber–Schauder system on the space $\mathcal{C}_{[0,1]}$ and study their properties. Note that, the latter basis corresponds to the interpolation of continuous functions by polygonal lines. We give the explicit form of the eigen projectors of these time operators and characterize the functions from their domains.

The plan of the paper is as follows. In Sect. 2 it is established a unified approach to the construction of time operators on arbitrary Banach spaces. This general approach is illustrated by determining the time operator for a simple dyadic map and connecting it with wavelets. Section 3 is devoted to a time operator associated with the Haar basis. It is shown here a number of results concerning the domain of time operator functions and the influence of time scalings on the underlying dynamics. In Sect. 4 it is constructed the time operator associated with the Faber–Schauder system on $\mathcal{C}_{[0,1]}$. The domain of this time operator as well as the domains of its scalings are also characterized. Some of the results from Sect. 3 have been already announced without proofs in [2].

2 Time Operators on Banach Spaces

Let V and T be two linear operators on a Banach space $\mathcal{B}$ such that V is bounded and T is densely defined. T is said to be a time operator associated with V if V preserves the domain of T, i.e. $V(D(T)) \subset D(T)$, and

$$TV^k = V^kT + kV^k, \quad \text{for } k \in \mathcal{I}, \tag{4}$$

where $\mathcal{I}$ is either the set $\mathbb{Z}$ of integers or the set $\mathbb{N}$ of natural numbers. This corresponds to the case where V is invertible or not respectively.

In the above definition of time operator, which is a straightforward generalization of (1), the operator V is interpreted as a generalized dynamics. In the sequel we will only consider a particular class of operators V that is specified below.

Consider a Banach space $\mathcal{B}$ decomposed as an infinite direct sum of closed subspaces

$$\mathcal{B} = \bigoplus_{n\in\mathcal{I}} \mathcal{B}_n \tag{5}$$

in the sense that each $x \in \mathcal{B}$ has a unique representation

$$x = \sum_{n\in\mathcal{I}} x_n, \tag{6}$$

where $x_n \in \mathcal{B}_n$ and the series (6) converges in $\mathcal{B}$. A linear operator V on the Banach space $\mathcal{B}$ of the form (5) will be called a *generalized shift* with respect to $\{\mathcal{B}_n\}$ if V is bounded and satisfies

$$V\mathcal{B}_n \subset \mathcal{B}_{n+1}, \quad \text{for } n \in \mathcal{I}. \tag{7}$$

We do not assume that V is an isometry neither that it maps $\mathcal{B}_n$ onto $\mathcal{B}_{n+1}$.

Let P_n be the projection from $\mathcal{B}$ onto $\mathcal{B}_n$, i.e. P_n is a linear operator on $\mathcal{B}$ that corresponds to each $x \in \mathcal{B}$ its n-th component x_n in representation (6). The family $\{P_n\}_{n\in\mathcal{I}}$ is a resolution of identity, i.e. $x = \sum_{n\in\mathcal{I}} P_n x$, for $x \in \mathcal{B}$, and determines a time operator. Namely we have

Proposition 1 *Assume that the Banach space $\mathcal{B}$ has the direct sum decomposition* (5) *and let $\{P_n\}_{n\in\mathcal{I}}$ be the corresponding family of projectors. Then the operator*

$$T = \sum_{n\in\mathcal{I}} nP_n, \tag{8}$$

defined for all $x \in \mathcal{B}$, for which the above series converges, is a time operator associated with an arbitrary generalized shift V with respect to $\{\mathcal{B}_n\}_{n\in\mathcal{I}}$.

Proof Let us show first that $V(D(T)) \subset D(T)$. Suppose that $x = \sum_n x_n$ belongs to the domain $D(T)$ of T. This means that the series $\sum_n P_n x = \sum_n n x_n$ converges in $\mathcal{B}$. On the other hand, the series $\sum_n x_n$ is also convergent. Since V is bounded, by the assumption, both series $\sum_n Vx_n$ and $\sum_n nVx_n$ converge. Thus $\sum_n (n+1)Vx_n = \sum_n nVx_n + \sum_n Vx_n$ is also convergent, which shows that $Vx \in D(T)$.

In order to show the identity (4) nottice first that

$$VP_n = P_{n+1}V, \quad \text{for each } n \in \mathcal{I}.$$

Indeed, if $x \in \mathcal{B}$, $x = \sum_n x_n$, then $VP_n x = Vx_n$. Conversely, $P_{n+1}Vx = P_{n+1}\sum_k Vx_k = Vx_n$, since Vx_n belongs to $\mathcal{B}_{n+1}$.

By the induction

$$V^k P_n = P_{n+k}V^k, \quad \text{for all } k, n \in \mathcal{I}. \tag{9}$$

Since the operators V^k are bounded and preserve the domain of T, it follows from (9) that

$$TV^k x = \sum_n nP_n V^k x = V^k \sum_n nP_{n-k}x = V^k \sum_n (n+k)P_n x = V^k T + kV^k x,$$

where we put $P_j = 0$ for $j \leq 0$. Note that if $\mathcal{I} = \mathbb{N}$, then $P_j V^k = 0$, for $j \leq k$. □

The direct sum decomposition (5) of the Banach space $\mathcal{B}$ determines a nested family of subspaces of $\mathcal{B}$ that is a counterpart of a multiresolution analysis in wavelets or a filtration in stochastic processes. Define

$$\mathcal{B}_{\leq n} \stackrel{\text{df}}{=} \bigoplus_{j\leq n} \mathcal{B}_j$$

and denote by E_n the projection from $\mathcal{B}$ onto $\mathcal{B}_{\leq n}$, i.e. $E_n = \sum_{j\leq n} P_j$. E_n is the projection onto the past up to the time instant n. The following elementary lemma relates the projectors E_n with the dynamics V.

Lemma 1 *Suppose that V is a generalized shift on $\mathcal{B} = \bigoplus_{n\in\mathcal{I}} \mathcal{B}_n$. Then*

$$V^k E_n = E_{n+k} V^k, \quad \text{for all } k, n \in \mathcal{I}. \tag{10}$$

Proof It is enough to show that (9) ⇒ (10), which is elementary in the case $\mathcal{I} = \mathbb{Z}$. If $\mathcal{I} = \mathbb{N}$ then we have

$$V^k E_n = V^k P_1 + \cdots + V^k P_n = (P_{1+k} + \cdots + P_{n+k}) V^k = (E_{n+k} - E_k) V^k = E_{n+k} V^k.$$

The latter equality follows from the direct sum decomposition (5) and the fact that $E_k V^k = 0$, for each $k \geq 1$. □

The above introduced concepts of time operator and generalized shift on Banach spaces, together with Proposition 1, allow to extend significantly the range of applications of the time operator method. As an example let us consider the time operator of the Renyi map which has been introduced in [1]. The 2-adic Renyi map defined on the unit interval [0, 1) by the formula

$$Sx = 2x \ (\mathrm{mod}\, 1).$$

is the simplest chaotic system and the prototype of exact endomorphisms [11]. The map S is invariant with respect to the Lebesgue measure. Its Koopman operator V

$$Vf(x) = f(Sx) = \begin{cases} f(2x), & \text{for } x \in [0, \frac{1}{2}), \\ f(2x-1), & \text{for } x \in [\frac{1}{2}, 1), \end{cases} \tag{11}$$

determines the evolution semigroup $\{V^n\}_{n\geq 0}$ on each space $L^p_{[0,1]}$, $p \geq 1$.

In order to determine a time operator associated with the Renyi map let us consider the Walsh-Paley system [20] $w_0, w_1, \ldots$ that form a Schauder basis in each of the spaces $L^p_{[0,1]}$, $1 < p < \infty$. This means that each function $f \in L^p_{[0,1]}$ has a unique expansion

$$f = w_0 + \sum_{j=1}^{\infty} a_j w_j = w_0 + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} a_k w_k$$

convergent in the L^p-norm.

Note that the functions $w_0, \ldots, w_{2^n-1}$ form a basis in the vector space of all functions that are measurable with respect to the σ-algebra $\mathcal{A}_n$ generated by the dyadic division of [0, 1] on 2^n parts. The block $w_{2^n}, \ldots, w_{2^{n+1}-1}$ is the contribution that is necessary to obtain all $\mathcal{A}_{n+1}$ measurable functions.

For each p, $1 < p < \infty$, we have the following direct sum decomposition

$$L^p_{[0,1]} = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots,$$

where $\mathcal{W}_0$ is the space of constant functions and $\mathcal{W}_n$, $n = 1, 2, \ldots$, is the linear space spanned by $w_{2^{n-1}}, \ldots, w_{2^n-1}$.

Denote by P_n the projection onto $\mathcal{W}_n$

$$P_n = \sum_{k=2^{n-1}}^{2^n-1} \langle \cdot, w_k \rangle w_k, \tag{12}$$

and put $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots$. We have

Proposition 2 ([1]) *The operator T defined on $\mathcal{W}$ as*

$$Tf = \sum_{n=1}^{\infty} n P_n f$$

is a time operator with respect to the semigroup $\{V^n\}_{n=1}^{\infty}$ generated by the Koopman operator of the Renyi map. Each number $n = 1, 2, \ldots$ is an eigenvalue of T and the functions $w_{2^{n-1}}, \ldots, w_{2^n-1}$ are the corresponding eigenvectors.

In order to prove the above proposition it is enough to notice that the Koopman operator V maps any Walsh function w_k from $\mathcal{W}_n$ on a Walsh function from $\mathcal{W}_{n+1}$, and to apply Proposition 1.

It was shown in [1] that the structure of T when restricted to the L^2-space coincides with the multiresolution analysis (MRA) associated with the Haar wavelet restricted to [0, 1]. The eigenspaces $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of the time operator of the Renyi map coincide with the corresponding wavelet spaces. The ladder of spaces $\mathcal{W}_0 \subset \mathcal{W}_0 \oplus \mathcal{W}_1 \subset \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \subset \cdots$ forms the multiresolution spaces of the Haar wavelet. This means that the time operator method is a straightforward generalization of MRA.

3 Time Operator Associated with the Haar Basis

The use of the Walsh basis for the construction of the time operator T associated with the Renyi map makes its spectral decomposition particularly simple. The Walsh functions are the eigenvectors of T and the action of the Koopman operator V is nothing but a shift from one Walsh function to another. However, the Walsh basis is not so convenient when dealing with continuous functions. For example, it is well known that the Haar expansion of a continuous function on the interval [0, 1] converges uniformly, while its Walsh series may be pointwise divergent. For such reasons it is in some cases more convenient to represent T in the basis of the Haar functions even if the action of V is no longer a simple shift of eigenvectors of T.

The representation of the time operator T associated with the Renyi map in the Haar basis has been already considered in [1]. This decomposition will be now considered in detail. It will be shown, generalizing the result from [1] that T, when represented in the Haar basis, is in fact a time operator with respect to a wider class of dynamical semigroups. A special attention will be paid to T as an operator on the space of continuous functions. It will be discussed the conditions under which a continuous function belongs to the domain of T and how time rescalings affect the dynamics by giving an estimation of the correlation function associated with the underlying dynamical semigroup. It turns out that for a wide class of functions the decay is exponential but the exponent depends on the "degree of smoothness" of the considered function.

Recall that the Haar functions χ_j on the interval [0, 1) are defined as follows:

$$\chi_1 \equiv 1, \qquad \chi_{2^n+k}(x) = 2^{\frac{n}{2}} \mathbb{1}_{[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}})}(x) - 2^{\frac{n}{2}} \mathbb{1}_{[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}})}(x),$$

for $n = 0, 1, \dots, k = 1, \dots, 2^n$.

Each function $f \in L^2_{[0,1]}$ has the expansion in the Haar basis, which can be written in one of the following equivalent forms

$$f = \sum_{j=1}^{\infty} a_j \chi_j = a_1 \chi_1 + \sum_{m=0}^{\infty} \sum_{k=1}^{2^m} a_{2^m+k} \chi_{2^m+k} = a_1 \chi_1 + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} a_k \chi_k, \tag{13}$$

with $a_j = \int_0^1 f(x)\chi_j(x)dx$.

The Haar functions form an orthonormal basis in $L^2_{[0,1]}$ and the linear space generated by $\{\chi_k\}_{k=2^{n-1}+1}^{2^n}$ coincides with $\mathcal{W}_n$. Also the orthogonal projection onto the space generated by $\{\chi_k\}_{k=2^{n-1}+1}^{2^n}$ coincides with the orthogonal projection P_n as defined by (12) for the Walsh basis. Introduced in Sect. 2 time operator T on $\mathcal{W}$ assumes now the form

$$Tf = \sum_{n=1}^{\infty} n \sum_{k=2^{n-1}+1}^{2^n} a_k \chi_k, \tag{14}$$

where $f \in L^2_{[0,1]}$ with $\int_0^1 f(x)\,dx = 0$ and a_k are as in (13). Therefore for a given n the Haar functions χ_{2^n+k}, $k = 1, \dots, 2^n$, are the eigenfunctions of T corresponding to the same eigenvalue n (see [1]). The Koopman operator V of the Renyi map does not transport a Haar function corresponding to the eigenvalue n onto a single Haar function but onto a linear combination of two Haar functions corresponding to the eigenvalue $n+1$ (see [1]). Nevertheless the assumptions of Proposition 1 are satisfied so that the commutation relation (4) still holds. In fact we have the following slightly more general result:

Theorem 1 *The operator T defined on $\mathcal{W} = L^2_{[0,1]} \ominus \{1\}$ as*

$$Tf = \sum_{n=1}^{\infty} n P_n,$$

where

$$P_n f = \sum_{2^{n-1}+1}^{2^n} \left[\int_0^1 f(x)\chi_k(x)dx \right] \chi_k$$

is a time operator with respect to any semigroup $\{V^n\}_{n \geq 0}$, where V is a bounded operator on $\mathcal{W}$ such that $V(\mathcal{W}_n) \subset \mathcal{W}_{n+1}$, for each $n = 1, 2, \dots$.

One of the advantages of the choice of the Haar functions for the representation of T is that they are more suitable when dealing with continuous functions. The following proposition gives a sufficient condition that a continuous function belongs to the domain of T defined by (14).

Proposition 3 *Any function $f \in \mathcal{C}_{[0,1]}$ such that its modulus of continuity satisfies the property*

$$\int_0^1 \omega_f(x) \frac{\log x}{x} dx > -\infty$$

belongs to the domain of T.

The proof of the above proposition can be found in [1]. However it will also follow from a more general theorem which will be proved below.

Remind that the modulus of continuity ω_f [21] is defined by:

$$\omega_f(\delta) = \sup_{\substack{x,y\in[0,1]\\ |x-y|\le\delta}} |f(x) - f(y)|, \quad 0 \le \delta \le 1,$$

for any $f \in \mathcal{C}_{[0,1]}$.

If a function f is a Lipschitz function, i.e. there are constants $K > 0$ and $0 < \alpha \le 1$ such that $|f(x) - f(y)| \le K|x - y|^\alpha$, for each $x, y \in [0, 1]$, then $\omega_f(x) \le K|x|^\alpha$. It is therefore easy to see that a Lipschitz function with an exponent α, $0 < \alpha \le 1$, satisfies the assumption of Proposition 3 and, consequently, belongs to the domain of T expanded in the Haar basis.

It is worth noting that if the time operator T is expanded in terms of the Walsh basis then the sufficient condition that a continuous function belongs to its domain is more restrictive. In particular, in the class of Lipschitz functions only those with the exponent $\alpha > \frac{1}{2}$ belong to the domain of T. The proof of this fact, which is based on some estimations for the Walsh–Fourier coefficients, will be presented elsewhere.

Note that the family of Haar functions forms also a Schauder basis in the Banach space $L^p_{[0,1]}$, $1 \le p < \infty$ and this basis is unconditional if $p > 1$. This means that every function $f \in L^p_{[0,1]}$ has the representation (13) convergent in L^p-norm (unconditionally convergent if $p > 1$). The Walsh functions also form a Schauder basis in L^p, $1 < p < \infty$, but not in L^1 (see [7] and references therein).

Let us now consider one of the basic tools in the study of dynamical systems with the use of time operator—the time scaling. This corresponds to filtering in signal processing. The time scaling means that the time operator T is replaced by some of its operator function $\Lambda(T)$, where $\Lambda(\cdot)$ is a real valued function. The application of Λ on T may, of course, affect its domain. Below we present sufficient conditions under which a continuous function f belongs to the domain of $\Lambda(T)$.

Theorem 2 *Let*

$$Tf = \sum_{n=1}^{\infty} n \sum_{k=2^{n-1}-1}^{2^n} a_k \chi_k, \tag{15}$$

where $a_k = \int_0^1 f(x)\chi_k\, dx$, $\int_0^1 f(x)\, dx = 0$, and let Λ be a real function on $\mathbb{N}$. Suppose that $f \in \mathcal{C}_{[0,1]}$.

(1) *If the modulus of continuity ω_f of f satisfies*

$$\sum_{n=1}^{\infty} |\Lambda(n)|^2 \omega_f^2\left(\frac{1}{2^n}\right) < \infty,$$

then f belongs to the domain of $\Lambda(T)$, where $\Lambda(T)$ is considered as an operator on $L^2 \ominus [1]$.

(2) *If ω_f satisfies*

$$\sum_{n=1}^{\infty} |\Lambda(n)|\, \omega_f\left(\frac{1}{2^n}\right) < \infty$$

then the series expansion of $\Lambda(T)$ converges uniformly on $[0, 1]$.

In particular, each function f for which

$$\int_0^1 \omega_f(x) \frac{\log x}{x}\, dx > -\infty$$

belongs to the domain of T and the expansion (15) *of Tf converges uniformly on* $[0, 1]$.

Proof If $\int_0^1 f(x)\, dx = 0$, then f has the following expansion in the Haar basis:

$$f = \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} a_k \chi_k, \tag{16}$$

where $a_n = \int_0^1 f(x)\chi_k(x)\, dx$, and the series converges in L^2-norm. Since f is continuous, the series (16) converges also uniformly on $[0, 1]$.

Formally, we have

$$\Lambda(T) f = \sum_{n=1}^{\infty} \Lambda(n) \sum_{k=2^{n-1}+1}^{2^n} a_k \chi_k. \tag{17}$$

Thus f belongs to the domain of $\Lambda(T)$ if and only if this series is L^2-convergent. Since $\{\chi_n\}$ is a complete orthonormal system in $L^2_{[0,1]}$, the convergence of (17) is in turn equivalent to

$$\sum_{n=1}^{\infty} \Lambda^2(n) \sum_{k=2^{n-1}+1}^{2^n} |a_k|^2 < \infty.$$

Because $|a_k| \le \frac{1}{2\cdot 2^{\frac{n-1}{2}}} \omega_f(\frac{1}{2^n})$ [4], we have

$$\begin{aligned}\sum_{n=1}^{\infty} \Lambda^2(n) \sum_{k=2^{n-1}+1}^{2^n} |a_k|^2 &\le \sum_{n=1}^{\infty} \Lambda^2(n) 2^{n-1} \frac{1}{4\cdot 2^{n-1}} \omega_f^2\left(\frac{1}{2^n}\right)\\ &\le \frac{1}{4} \sum_{n=1}^{\infty} \Lambda^2(n) \omega_f^2\left(\frac{1}{2^n}\right),\end{aligned}$$

which proves (1). In a similar way (note that $\chi_{2^{n-1}+1}, \dots, \chi_{2^n}$ have disjoint supports) we obtain

$$\sum_{n=1}^{\infty} |\Lambda(n)| \sum_{k=2^{n-1}+1}^{2^n} |a_k||\chi_k| \le \frac{1}{2} \sum_{n=1}^{\infty} |\Lambda(n)| \omega_f\left(\frac{1}{2^n}\right),$$

which proves (2).

In order to prove the last assertion take $\Lambda(n) = n$ and notice that the function is non-decreasing on [0, 1] and $-\frac{\log t}{t}$ is decreasing, we have

$$\begin{aligned}
-\int_0^1 \omega_f(t)\frac{\log t}{t}dt &= \sum_{n=1}^{\infty}\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \omega_f(t)\left(-\frac{\log t}{t}\right)dt \\
&\geq \sum_{n=1}^{\infty}\omega_f\left(\frac{1}{2^{n-1}}\right)\left(-\frac{\log\frac{1}{2^n}}{\frac{1}{2^n}}\right)\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right) \\
&= \log 2\sum_{n=1}^{\infty} n\omega_f\left(\frac{1}{2^{n-1}}\right) \\
&\geq 4^{-1}\log 2\sum_{n=1}^{\infty} n\omega_f\left(\frac{1}{2^n}\right). \qquad (18)
\end{aligned}$$

The last inequality is a consequence of the property $\omega_f(x+y) \leq \omega_f(x) + \omega_f(y)$ valid for $x, y, x+y \in [0, 1]$. Since the left hand side of (18) is finite by the assumption, the series (15) is uniformly and absolutely convergent. This concludes the proof. □

Finally we shall show how scaling affect the dynamics. We consider the action semigroup $\{V^N\}$ on the functions transformed through $\Lambda(T)$. The relevant value that we would like to evaluate is the correlation function

$$R_f(N) \stackrel{\text{df}}{=} (V^N\Lambda(T)(f), \Lambda(T)(f)),$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2_{[0,1]}$.

Theorem 3 *Let V be the Koopman operator of some map of the interval* [0, 1] *such that the corresponding semigroup $\{V^N\}$ on $L^2_{[0,1]}$ satisfies the assumptions of Theorem* 1. *Then for every function $f \in \mathcal{C}_{[0,1]}$ which belongs to the domain of $\Lambda(T)$ we have*

$$|R_f(N)| \leq \frac{1}{4}\sum_{n=1}^{\infty}|\Lambda(n)\Lambda(n+N)|\omega\left(\frac{1}{2^n}\right)\omega\left(\frac{1}{2^{n+N}}\right).$$

Proof In order to calculate the correlation function let us represent Λ as follows

$$\Lambda(f) = \sum_{n=0}^{\infty}\Lambda(n+1)\sum_{k=1}^{2^n} a_{2^n+k}\chi_{2^n+k}.$$

Let N and n be fixed. By the assumption each $V^N\chi_{2^n+k}$, $k = 1,\ldots,2^n$, is a linear combination of some basis elements $\chi_{2^{n+N}+k}$:

$$V^N\chi_{2^n+k} = \sum_{j=1}^{n_k}\alpha_{2^{n+N}+l_j(k)}\chi_{2^{n+N}+l_j(k)}, \qquad k = 1,\ldots,2^n, \qquad (19)$$

for some choice of indices $l_1(k),\ldots,l_{n_k}(k)$.

Since V is also a Koopman operator, V^N is a multiplicative map, i.e. $Vfg = VfVg$. Because, for a given n, the functions χ_{2^n+k}, $k = 1, \dots, 2^n$, have disjoint supports, then also $V^N \chi_{2^n+k}$ must have disjont supports. This implies that in the representation (19) all $l_j(k)$ are different and

$$n_1 + \cdots + n_{2^n} \le 2^{n+N}. \tag{20}$$

We have

$$\begin{aligned}
R_f(N) &= \left(\sum_{n=0}^{\infty} \Lambda(n+1) \sum_{k=1}^{2^n} a_{2^n+k} V^N \chi_{2^n+k}, \sum_{n=0}^{\infty} \Lambda(n+1) \sum_{k=1}^{2^n} a_{2^n+k} \chi_{2^n+k} \right) \\
&= \sum_{n=0}^{\infty} \Lambda(n+1)\Lambda(n+N+1) \\
&\quad \times \left(\sum_{k=1}^{2^n} a_{2^n+k} \sum_{j=1}^{n_k} \alpha_{2^{n+N}+l_j(k)} \chi_{2^{n+N}+l_j(k)}, \sum_{k=1}^{2^{n+N+1}} a_{2^{n+N}+k} \chi_{2^{n+N}+k} \right) \\
&= \sum_{n=0}^{\infty} \Lambda(n+1)\Lambda(n+N+1) \sum_{k=1}^{2^n} a_{2^n+k} \sum_{j=1}^{n_k} a_{2^{n+N}+l_j(k)} \alpha_{2^{n+N}+l_j(k)}.
\end{aligned}$$

Thus

$$\begin{aligned}
|R_f(N)| &\le \sum_{n=0}^{\infty} |\Lambda(n+1)\Lambda(n+N+1)| \sum_{k=1}^{2^n} \frac{1}{2 \cdot 2^{\frac{n}{2}}} \omega\left(\frac{1}{2^{n+1}}\right) \\
&\quad \times \sum_{j=1}^{n_k} \frac{1}{2 \cdot 2^{\frac{n+N}{2}}} \omega\left(\frac{1}{2^{n+N+1}}\right) |\alpha_{2^{n+N}+l_j(k)}| \\
&= \frac{1}{4} \cdot \frac{1}{2^{\frac{N}{2}}} \sum_{n=0}^{\infty} |\Lambda(n+1)\Lambda(n+N+1)| \frac{1}{2^n} \omega\left(\frac{1}{2^{n+1}}\right) \omega\left(\frac{1}{2^{n+N+1}}\right) \\
&\quad \times \sum_{k=1}^{2^n} \sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)}|.
\end{aligned}$$

Note that

$$\|V^N \chi_{2^n+k}\|_{L^2}^2 = \left\| \sum_{j=1}^{n_k} \alpha_{2^{n+N}+l_j(k)} \chi_{2^{n+N}+l_j(k)} \right\|^2 = \sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)}|^2,$$

and since $\|V\| \le 1$, we have

$$\sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)}|^2 \le \|V^N\|^2 \|\chi_{2^{n+N}+k}\|_{L^2}^2 \le 1.$$

Therefore, applying Hölder's inequality we get

$$\sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)^{\frac{1}{2}}}| \leq \sqrt{n_k}\left(\sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)}|^2\right)^{\frac{1}{2}} \leq \sqrt{n_k}.$$

Applying Hölder's inequality once more together with (20) we obtain

$$\sum_{k=1}^{2^n}\sum_{j=1}^{n_k} |\alpha_{2^{n+N}+l_j(k)}| \leq \sum_{k=1}^{2^n}\sqrt{n_k} \leq 2^{\frac{n}{2}}\left(\sum_{k=1}^{2^n} n_k\right)^{\frac{1}{2}} \leq 2^{\frac{n}{2}} \cdot 2^{\frac{n+N}{2}}.$$

Consequently

$$|R_f(N)| \leq \frac{1}{4}\cdot\frac{1}{2^{\frac{N}{2}}}\sum_{n=0}^{\infty} |\Lambda(n+1)\Lambda(n+N+1)| \frac{1}{2^n}\omega\left(\frac{1}{2^{n+1}}\right)\omega\left(\frac{1}{2^{n+N+1}}\right)2^n 2^{\frac{N}{2}},$$

and rearranging the summation we finally get

$$|R_f(N)| \leq \frac{1}{4}\sum_{n=1}^{\infty} |\Lambda(n)\Lambda(n+N)|\omega\left(\frac{1}{2^n}\right)\omega\left(\frac{1}{2^{n+N}}\right),$$

which ends the proof. □

Under the same assumptions as in Theorem 3 we have

Corollary 1 *If f is a Hölder function with the exponent p, $0 < p \leq 1$, then*

$$|R_f(N)| \leq \frac{1}{4}\cdot\frac{1}{2^{pN}}\sum_{n=1}^{\infty} |\Lambda(n)\Lambda(n+N)| \frac{1}{2^{2pn}}. \tag{21}$$

Corollary 2 *If $\Lambda(x)$ is a bounded function then the series on the right hand side of (21) is convergent and*

$$|R_f(N)| \leq K\left(\frac{1}{2^p}\right)^N = Ke^{-(p\ln 2)N},$$

for some $K > 0$.

It should be noticed that for a bounded Λ any continuous function is in the domain of $\Lambda(T)$. In particular, taking $\Lambda(x) \equiv 1$ we see that the decay of the correlations of $(V^N f, f)$ is at least exponential with the exponent depending on the degree of smoothness of f. If f is differentiable then the exponent is $\ln 2$.

Example Let V be the Koopman operator of the Renyi map. Since

$$V\chi_{2^n+k} = \frac{1}{\sqrt{2}}(\chi_{2^{n+1}+k} + \chi_{2^{n+1}++2^n+k}),$$

V satisfies the assumptions of Theorem 1.

4 Time Operator Associated with the Faber–Schauder Basis in $\mathcal{C}_{[0,1]}$

Although each continuous function can be expanded in terms of the Haar basis the Haar functions lay outside the space $\mathcal{C}_{[0,1]}$. We shall show that the Haar basis in $L^1_{[0,1]}$ can be transported to $\mathcal{C}_{[0,1]}$ by the means of integration giving rise to a new basis in $\mathcal{C}_{[0,1]}$. Namely, let us define the operator of indefinite integration $J : L^1_{[0,1]} \to \mathcal{C}_{[0,1]}$:

$$(Jf)(t) \overset{\mathrm{df}}{=} \int_0^t f(s)ds, \quad \text{for } f \in L^1_{[0,1]}.$$

The range of J consists of absolutely continuous functions. Since the series (13) is also uniformly convergent [10] we can apply J to both sides getting

$$\begin{aligned}(Jf)(t) &= \sum_{j=1}^{\infty}\left[\int_0^1 f(s)\chi_j(s)ds\right](J\chi_j)(t)\\ &= \sum_{j=1}^{\infty}\left[\int_0^1 \chi_j(s)d(Jf)(s)\right]\varphi_j(t),\end{aligned} \tag{22}$$

where $\varphi_j(t) \overset{\mathrm{df}}{=} (J\chi_j)(t)$, $j = 1, 2, \dots$.

Actually the representation (22) extends on all functions $g \in \mathcal{C}_{[0,1]}$. To be more precise the family $\{\varphi_j\}_{j=1}^{\infty}$ together with the constant function $\varphi_0 \equiv 1$ form a Schauder basis in the Banach space $\mathcal{C}_{[0,1]}$. We have [4]

$$g(t) = g(0)\varphi_0 + \sum_{j=1}^{\infty}\left[\int_0^1 \chi_j(s)\,dg(s)\right]\varphi_j(t), \tag{23}$$

where the series converges uniformly in [0, 1].

In a similar way, applying J to both sides of (14), we can transport the time operator T to $\mathcal{C}_{[0,1]}$. To be more precise, let $\widetilde{\mathcal{C}}$ be the space of all functions $g \in \mathcal{C}_{[0,1]}$ such that $g(0) = g(1) = 0$ and let $\mathcal{C}_n$, $n = 1, 2, \dots$, be the subspace of $\widetilde{\mathcal{C}}$ spanned by φ_k, $2^{n-1} < k \le 2^n$. Define the operator $\widetilde{P}_n : \widetilde{\mathcal{C}} \to \mathcal{C}_n$ putting

$$\widetilde{P}_n g(t) = \sum_{k=2^{n-1}+1}^{2^n} \int_0^1 \chi_k(s)dg(s)\varphi_k(t). \tag{24}$$

Theorem 4 *The operator $\widetilde{T}$ defined on $\widetilde{\mathcal{C}}$ as*

$$\widetilde{T} = \sum_{n=1}^{\infty} n\widetilde{P}_n,$$

is a time operator with respect to any semigroup $\{V^n\}_{n\ge 0}$, where V is a bounded operator on $\widetilde{\mathcal{C}}$ such that $V(\mathcal{C}_n) \subset \mathcal{C}_{n+1}$, for each $n = 1, 2, \dots$. The explicit form of $\widetilde{P}_n$ is

$$\widetilde{P}_n g(t) = 2^{\frac{n-1}{2}} \sum_{k=1}^{2^{n-1}}\left[2g\left(\frac{2k-1}{2^n}\right) - g\left(\frac{k-1}{2^{n-1}}\right) - g\left(\frac{k}{2^{n-1}}\right)\right]\varphi_{2^{n-1}+k}(t) \tag{25}$$

Proof Since the functions $\varphi_0, \varphi_1, \dots$ form a Schauder basis in $\mathcal{C}_{[0,1]}$ we have the direct sum decomposition

$$\widetilde{\mathcal{C}} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$$

and each $\widetilde{P}_n$ is the projector onto $\mathcal{C}_n$. It is easy to see that

$$V\widetilde{P}_n = \widetilde{P}_{n+1}V, \tag{26}$$

for each $n \in \mathbb{N}$. Indeed, since each $g \in \widetilde{\mathcal{C}}$ has a unique expansion

$$g = \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} b_k \varphi_k. \tag{27}$$

then

$$V\widetilde{P}_n g = V\left(\sum_{k=2^{n-1}+1}^{2^n} b_k \varphi_k\right) = \sum_{k=2^{n-1}+1}^{2^n} b_k V\varphi_k.$$

On the other hand, among the elements $\{V\varphi_j\}$ only those with $2^{n-1} < j \le 2^n$ are elements of $\mathcal{C}_{n+1}$. Therefore

$$\widetilde{P}_{n+1}Vg = \sum_{j=2^{n-1}+1}^{2^n} b_j V\varphi_j,$$

which implies (26). Consequently the assumptions of Proposition 1 are satisfied and $\widetilde{T}$ is a time operator with respect to $\{V^n\}_{n\ge 0}$. The explicit form (25) of $\widetilde{P}_n$ follows directly from (24). □

Constructed in this way time operator arises as the integral transformation of the time operator T for the Renyi map expanded in terms of the Haar basis, i.e. for $g = Jf$ we have

$$\widetilde{T}g = Tf.$$

Moreover, since $\widetilde{T}$ is a time operator with respect to any bounded operator V, which maps each Schauder function $\varphi_{2^{n-1}+k}$, $k = 1, \dots, 2^{n-1}$, onto a linear combination of the functions $\varphi_{2^n+k'}$, $k' = 1, \dots, 2^n$, it can be also associated with the Koopman operator V of the Renyi map (11) acting on the space $\widetilde{\mathcal{C}}$. Indeed, V is, of course, bounded on $\widetilde{\mathcal{C}}$ and one can check easily that

$$V\varphi_{2^{n-1}+k} = \sqrt{2}(\varphi_{2^n+k} + \varphi_{2^n+2^{n-1}+k}),$$

for $n = 1, 2, \dots$, $k = 1, \dots, 2^n$.

Observe also that the Koopman operator V of the Renyi map together with the integral operator J satisfy the following commutation relation

$$VJf = 2JVf, \tag{28}$$

valid for each $f \in L^1$ such that $\int_0^1 f(s)d(s) = 0$.

Using the time operator terminology we can say now that the function $g \in \widetilde{\mathcal{C}}$, has the age n if its representation (23) consists of the n-th block, i.e. those with the indices $k = 2^{n-1} + 1, \dots, 2^n$. Therefore the flow of time means step by step interpolation of g by polygonal lines. The polygonal line $l_n(x)$, $n = 1, 2, \dots$, corresponding to the dyadic division of the interval [0, 1] on 2^n parts is

$$l_n(x) = \sum_{k=1}^{2^n} \left\{ \left[g\left(\frac{k}{2^n}\right) - g\left(\frac{k-1}{2^n}\right) \right] (2^n x - k + 1) + g\left(\frac{k-1}{2^n}\right) \right\} \mathbb{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n}]}(x).$$

Since the time operators considered here are defined on infinite dimensional Banach spaces, it is easy to see that their domains are always proper subsets of the underlined spaces. Therefore it arises the problem of characterization of the domain of a time operator. It is also important for applications of time operator techniques, especially for filtering, to characterize the domain of a function of a given time operator.

It is easy to see that if the eigenfunctions of a time operator T defined on a Banach space $\mathcal{B}$ form an unconditional Schauder basis then for any bounded function $\Lambda : \mathbb{N} \to \mathbb{R}$ the domain of $\Lambda(T)$ coincides with $\mathcal{B}$. However the space $\mathcal{C}_{[0,1]}$ does not have an unconditional basis. Therefore not for each bounded function Λ the operator $\Lambda(\widetilde{T})$ is correctly defined on $\mathcal{C}_{[0,1]}$. The next theorem provides sufficient conditions for a function g to be in the domain of $\widetilde{T}$, as well as to be in the domain of $\Lambda(\widetilde{T})$ in terms of the modulus of continuity.

Theorem 5 *Let Λ be a real valued function defined on $\mathbb{N}$. Any function $g \in \widetilde{\mathcal{C}}$ such that its modulus of continuity ω_g satisfies the property*

$$\sum_{n=1}^{\infty} |\Lambda(n)| \omega_g(2^{-n}) < \infty \tag{29}$$

belongs to the domain of $\Lambda(\widetilde{T})$ and the series $\sum_n \Lambda(n) \widetilde{P}_n g(t)$ is uniformly and absolutely convergent. In particular, if the modulus of continuity ω_g satisfies

$$\int_0^1 \omega_g(t) \frac{\log t}{t} dt > -\infty \tag{30}$$

then g belongs to the domain of $\widetilde{T}$ and the series

$$\sum_{n=1}^{\infty} n \widetilde{P}_n g(t) \tag{31}$$

is uniformly and absolutely convergent.

Proof Let $g \in \widetilde{\mathcal{C}}$ be such that its modulus of continuity satisfies (29). Let $b_{n,k} \stackrel{\mathrm{df}}{=} \int_0^1 \chi_{2^{n-1}+k}(s) dg(s)$. We have

$$\begin{aligned} |b_{n,k}| &= \left| \int_0^1 \chi_{2^{n-1}+k}(s) dg(s) \right| \\ &\le 2^{\frac{n-1}{2}} \left| g\left(\frac{k-1}{2^{n-1}}\right) - g\left(\frac{2k-1}{2^n}\right) \right| + 2^{\frac{n-1}{2}} \left| g\left(\frac{2k-1}{2^n}\right) - g\left(\frac{k}{2^{n-1}}\right) \right| \\ &\le 2^{\frac{n+1}{2}} \omega_g\left(\frac{1}{2^n}\right), \end{aligned}$$

for $n = 1, 2, \ldots,\ k = 1, \ldots, 2^{n-1}$. Therefore

$$\sum_{n=1}^{\infty} |\Lambda(n)\widetilde{P}_n g(t)| \le \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} |\Lambda(n) b_{n,k}|\varphi_{n,k}(t) \le 2\sum_{n=1}^{\infty} |\Lambda(n)|\omega_g\left(\frac{1}{2^n}\right)$$

(nottice that for a fixed n the supports of $\varphi_{n,k},\ k = 1, \ldots, 2^{n-1}$ are disjoint), which proves the first part of the theorem. To show the second part observe that putting $\Lambda(x) = x$ we have

$$\sum_{n=1}^{\infty} n|\widetilde{P}_n g(t)| \le \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} n|b_{n,k}|\varphi_{n,k}(t) \le \sum_{n=1}^{\infty} n\omega_g\left(\frac{1}{2^n}\right).$$

On the other hand, since the function $\omega_g(t)$ is non-decreasing on [0, 1] and $-\frac{\log t}{t}$ is decreasing, we have

$$\begin{aligned}
-\int_0^1 \omega_g(t)\frac{\log t}{t}dt &= \sum_{n=1}^{\infty}\int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \omega_g(t)\left(-\frac{\log t}{t}\right)dt \\
&\ge \sum_{n=1}^{\infty}\omega_g\left(\frac{1}{2^{n-1}}\right)\left(-\frac{\log\frac{1}{2^n}}{\frac{1}{2^n}}\right)\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) \\
&= \log 2\sum_{n=1}^{\infty} n\omega_g\left(\frac{1}{2^{n-1}}\right) \\
&\ge 4^{-1}\log 2\sum_{n=1}^{\infty} n\omega_g\left(\frac{1}{2^n}\right).
\end{aligned} \tag{32}$$

The last inequality is a consequence of the property $\omega_g(x+y) \le \omega_g(x) + \omega_g(y)$ valid for $x, y, x+y \in [0, 1]$. Since the left hand side of (32) is finite by the assumption, the series (31) is uniformly and absolutely convergent. This concludes the proof. □

Corollary 3 *If Λ is a bounded function on $\mathbb{N}$, then each $g \in \widetilde{C}$ such that $\sum_{n=1}^{\infty}\omega_g(2^{-n}) < \infty$ belongs to the domain of $\Lambda(\widetilde{T})$.*

Corollary 4 *If g is a Lipschitz function with an exponent $0 < p \le 1$, then g belongs to the domain of $\widetilde{T}$.*

Proof It follows from the definition of ω_g that if g satisfies $|g(x) - g(y)| \le |x - y|^p$ then $\omega_g(t) \le Kt^p$. Since $\int_0^1 t^{p-1}\log t\,dt > -\infty$, for $p > 0$, the condition (30) is satisfied. □

We have already mentioned about the importance of time scalings realized through the Λ operators defined as functions of the time operator. Somewhat different is the role of the integration transformation J, which satisfies together with T the commutation relation (28). Applying the transformation J on a functional basis makes approximations "smoother". We have seen already that applying J on the orthonormal Haar basis we obtain the time operator associated with approximations by continuous functions, i.e. with the Faber–Schauder basis in $\mathcal{C}_{[0,1]}$. Similarly, starting from a time operator associated with an orthogonal basis of continuous functions we can get a time operator associated with approximation in the space

$\mathcal{C}^{(1)}_{[0,1]}$ of differentiable functions. As an example let us consider the Franklin system ϕ_n, $n = 0, 1, \ldots$ in $\mathcal{C}_{[0,1]}$. Recall that the functions ϕ_n are obtained through the Schmidt orthonormalization of the Faber–Schauder functions [5, 6, 20]. The Franklin system is a Schauder basis in $\mathcal{C}_{[0,1]}$. We can therefore apply Proposition 1 constructing, as in Theorem 4, the time operator associated with a given partition of $\{\phi_n\}$ on blocks. On the other hand the system

$$1, J\phi_0, J\phi_1, \ldots \tag{33}$$

is also a Schauder basis in $\mathcal{C}_{[0,1]}$ and each $f \in \mathcal{C}_{[0,1]}$ has the expansion

$$f(t) = f(0) + \sum_{n=0}^{\infty} a_n J\phi_n,$$

where $a_n = \int_0^1 \phi_n(s) df(s)$. This implies that (33) is also a Schauder basis in $\mathcal{C}^{(1)}_{[0,1]}$ endowed with the norm $\|f\| \stackrel{\text{df}}{=} \max_{0\le s\le 1} |f(s)| + \max_{0\le s\le 1} |f'(s)|$. Repeating again the proof of Theorem 4 we obtain a time operator in $\mathcal{C}^{(1)}_{[0,1]}$ associated with this basis, which is nothing but the composition of the integral transformation with time operator constructed formerly on $\mathcal{C}_{[0,1]}$.

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